The Basic Concepts of Trigonometry
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Source: *Mathematics Magazine*, Vol. 22, No. 3 (Jan. - Feb., 1949), pp. 139-150
Published by: Mathematical Association of America
Stable URL: http://www.jstor.org/stable/3029686
Accessed: 22-11-2015 21:06 UTC

TEACHING OF MATHEMATICS

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THE BASIC CONCEPTS OF TRIGONOMETRY

Estelle Mazziotta

Trigonometry might well be called the science of getting from *here* to *there*, especially when *there* is inaccessible. The problem of going from *here* to *there* resolves itself into two fundamental components: How far is it, and in what direction? From this it can be seen that trigonometry deals with lines (which measure distance) and angles (which indicate direction). The lines and angles are formed into the simplest of polygons, the triangle, hence the name ''trigonometry'', which really means ''triangle measurement''.

This branch of mathematics is the tool of the surveyor who measures land and computes the heights of mountains without scaling them, and also of the navigator (both sea and air) who sets courses and determines speeds. Trigonometry is one of the oldest branches of science and is associated with the surveying of land in ancient Egypt as well as with the development of both astrology and astronomy. Early surveying instruments, the forerunners of the modern transit, testify to this association by their elaborate decoration with the signs of the zodiac and representations of various constellations.

One of the basic problems of trigonometry is that of determining the height of an inaccessible object. Suppose, for example, that we wish to determine the height of a tree without being obliged to climb it. We can measure a convenient distance, say 40 feet, from the foot of the tree and, at this point, sight the top of the tree so that the line of sight completes a triangle that looks like Fig. 1.

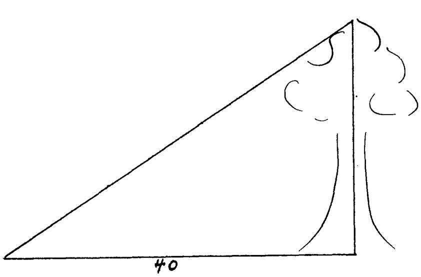

Figure 1

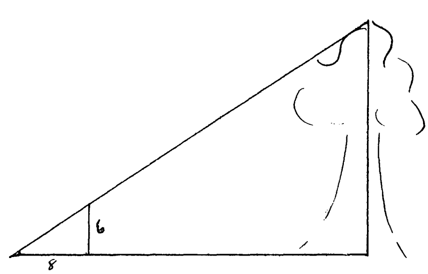

Figure 2

If a six-foot pole (see Fig. 2) is placed in such a way that its tip lies along the line of sight when its foot is perpendicular to the ground, we can compare the small triangle with the large and use this comparison to find the height of the tree, as we shall now show. Suppose the foot of the pole touches the ground at a distance of eight feet from the observer. It is apparent that the base of the small triangle is contained in the base of the larger one five times. Hence, if we fit a series of such small triangles along the line of sight as in Fig. 3, there would evidently be five of them. The base of the smaller triangle bears the same relationship to the total distance from the foot of the tree as the height of the pole bears to the height of the tree, that is, 8/40 is equal to 6/30.

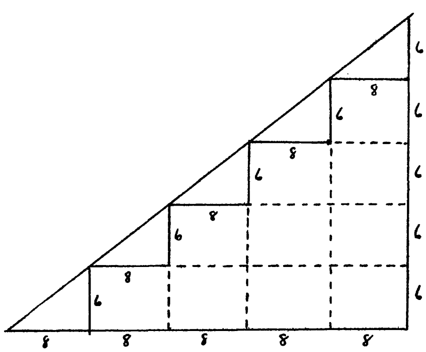

Figure 3

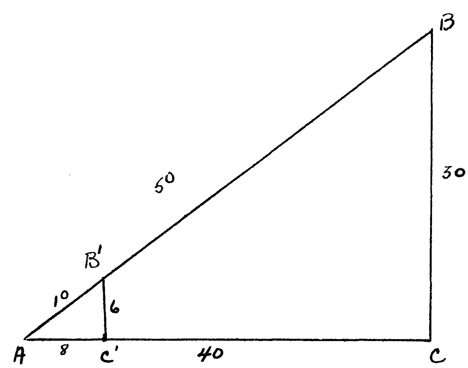

Figure 4

Such triangles are said to be similar, that is, alike in shape. *Their corresponding sides are proportional and their corresponding angles are equal.*

Our example serves as a key for understanding certain fundamental relations between the sides and angles of any right triangles which have a common acute angle. Consider the two triangles ABC and $AB'C'$ in Fig. 4. Each side of triangle ABC is exactly five times as large as the corresponding side of triangle $AB'C'$. Another way to show this relation is to equate corresponding ratios in the two triangles as follows:

$$\frac{AC'}{AB'} = \frac{AC}{AB} \qquad \frac{C'B'}{AB'} = \frac{CB}{AB} \qquad \frac{C'B'}{AC'} = \frac{CB}{AC}$$

or, in terms of the above example:

$$\frac{8}{10} = \frac{40}{50} \qquad \frac{6}{10} = \frac{30}{50} \qquad \frac{6}{8} = \frac{30}{40}$$

These ratios are not dependent upon the area of the right triangle; they are dependent only upon the angle A. That is, they are the same

for all such right triangles, $AB'C'$ and ABC, which have the common angle A. We call these ratios *Trigonometric functions* of angle A— trigonometric because they involve the sides of a triangle, functions of angle A because their values depend on the value of angle A. We name them as follows (see Fig. 5):

$$\frac{\text{side opposite}}{\text{hypotenuse}} = \frac{CB}{AB} = \text{sine of } A, \text{ written } \sin A$$

$$\frac{\text{side adjacent}}{\text{hypotenuse}} = \frac{AC}{AB} = \text{cosine of } A, \text{ written } \cos A$$

$$\frac{\text{side opposite}}{\text{side adjacent}} = \frac{CB}{AC} = \text{tangent of } A, \text{ written } \tan A.$$

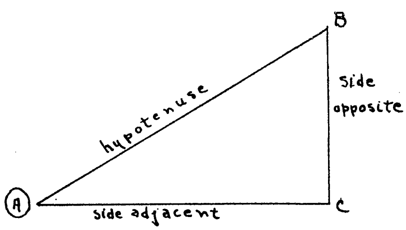

Figure 5

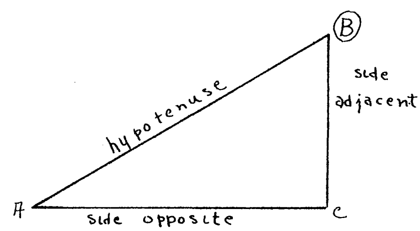

Figure 6

If we wish to list the trigonometric functions of angle B, we rename the sides of the right triangle with reference to B, as in Fig. 6. Then we find that $\sin B = \frac{AC}{AB}$, $\cos B = \frac{CB}{AB}$, and $\tan B = \frac{AC}{CB}$. Notice that the sine of A and the cosine of B are the same; likewise, the cosine of A and the sine of B are the same. Since A and B are the acute angles of a right triangle, their sum is a right angle, and they are said to be complementary. The word ''cosine'' implies ''complement's sine'', *i.e.*, the cosine of B is the sine of the complement of B, and vice versa.

It is possible, of course, to write three other ratios between the three sides of a right triangle taken in pairs. However, since these would be merely reciprocals of the ones already named, we mention them only for the convenience of the reader who may meet with them elsewhere. These additional functions are called cotangent, secant and cosecant and are respectively the reciprocals of the tangent, cosine and sine.

The variations in the values of the trigonometric functions of an angle which occur with changes in the angle can be demonstrated quite simply by placing the right triangle in a circle with radius one unit, in such a way that the radius appears in the denominators of the trigonometric ratios. Then we need observe only the changes which occur in single lines as the angle increases or decreases.

In Fig. 7, $AR = AP = AP' = 1$. In triangle APQ, $\sin A = \frac{PQ}{AQ} = \frac{PQ}{1}$; $\cos A = \frac{AP}{AQ} = \frac{AP}{1}$.

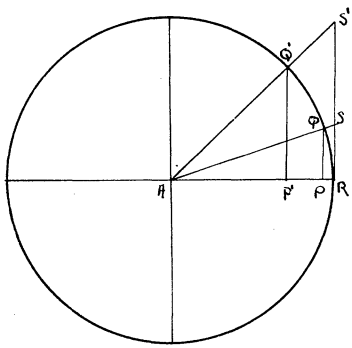

Figure 7

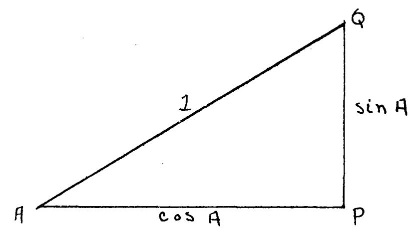

Figure 8

As angle A increases, PQ assumes a series of positions in the circle, one of which is shown by $P'Q'$. This indicates that the sine of an angle increases as the angle increases. It may also be seen from the diagram that sin A will reach a maximum value of 1 when A becomes 90°. Similarly, as angle A decreases PQ also decreases, and when A becomes 0° PQ reaches a minimum value of 0. In the case of the cosine, exactly the reverse is true. As angle A increases AP decreases, and when A reaches 90° the cosine, AP, becomes 0, and when A is 0° it will have a maximum value of 1. To observe corresponding changes in the tangent, it is necessary to consider another triangle, namely ARS. In this triangle, $\tan A = \frac{RS}{AR} = \frac{RS}{1}$. Like the sine, the tangent increases with the angle, but, as the figure shows, the increase is much more rapid. As the angle A approaches 90° this increase is without limit, and there is no tangent of 90° since this would involve division by 0.

The triangle in Fig. 8, which resembles the one in the unit circle, shows that the functions are so related to each other that given any one the other two can be determined. As an illustration we will find cos A and tan A in terms of sin A. Since we defined tan A as $\frac{\text{side opposite}}{\text{side adjacent}}$, evidently, $\tan A = \frac{\sin A}{\cos A}$. It follows also, from the Pythagorean theorem, (which states that the sum of the squares of the legs of a right triangle is equal to the square of the hypotenuse) that

$$\sin^2 A + \cos^2 A = 1.$$

From this equation

$$\cos^2 A = 1 - \sin^2 A$$

whence

$$\cos A = \sqrt{1 - \sin^2 A}$$

Substituting the latter in $\tan A = \dfrac{\sin A}{\cos A}$ gives

$$\tan A = \frac{\sin A}{\sqrt{1 - \sin^2 A}}$$

The *numerical values* of the functions of any angle could be approximated by means of a carefully constructed diagram, for instance, by measuring the lines PQ, AP and SR in Fig. 7. However, the values of the trigonometric functions are in general better approximated by means of series derived in other branches of mathematics. But the values of the functions of certain useful angles can be determined exactly from the properties of certain specific triangles. Let us consider, for example, an *isosceles right triangle,* that is, one having two equal legs. Suppose we denote the equal legs (Fig. 9) by any letter, say b; then we find the hypotenuse by the Pythagorean theorem: $c^2 = b^2 + b^2$, whence $c^2 = 2b^2$ and $c = b\sqrt{2}$.

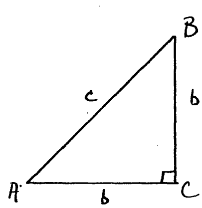

Figure 9

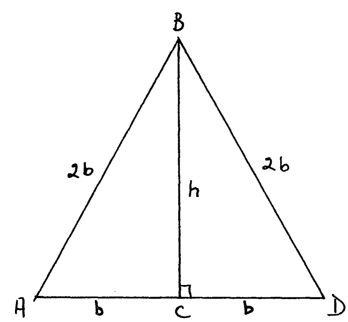

Figure 10

Since the sum of the angles of a triangle is 180^0 and the angles opposite the equal sides of a triangle are equal, each of the acute angles must be 45^0, so we can write the values of the functions of an angle of 45^0.

$$\sin 45^0 = \frac{b}{b\sqrt{2}} = \frac{1}{\sqrt{2}} = \frac{1\sqrt{2}}{\sqrt{2}\,\sqrt{2}} = \frac{\sqrt{2}}{2} = \frac{1.4142^+}{2} = 0.7071^+$$

$$\cos 45^0 = \frac{b}{b\sqrt{2}} = \frac{1}{\sqrt{2}} = 0.7071^+$$

$$\tan 45^0 = \frac{b}{b} = 1.0000$$

In like manner, we can make use of an equilateral triangle—one having all sides equal and each angle 60^0 – to find the values of the

functions of 30° and 60° (see Fig. 10). To avoid fractions, let us take each side of the triangle equal to $2b$ units in length. The perpendicular from B to the base bisects the base and the angle at B, producing right triangles having acute angles equal to 60° and 30°. The altitude, h, is found as follows:

$$h^2 + b^2 = (2b)^2$$

Transposing and combining like terms,

$$h^2 = 4b^2 - b^2 = 3b^2$$

Taking the square root of both sides,

$$h = b\sqrt{3}$$

Hence the functions have the following values:

$$\sin 30^\circ = \frac{b}{2b} = \frac{1}{2} = 0.5000$$

$$\cos 30^\circ = \frac{b\sqrt{3}}{2b} = \frac{1\sqrt{3}}{2} = \frac{\sqrt{3}}{2} \quad \frac{1.7321^-}{2} = 0.8660^+$$

$$\tan 30^\circ = \frac{b}{b\sqrt{3}} = \frac{1\sqrt{3}}{\sqrt{3}\sqrt{3}} = \frac{\sqrt{3}}{3} = \frac{1.7321^-}{3} = 0.5774^-$$

$$\sin 60^\circ = \cos 30^\circ = 0.8660^+$$

$$\cos 60^\circ = \sin 30^\circ = 0.5000$$

$$\tan 60^\circ = \frac{b\sqrt{3}}{b} = \frac{\sqrt{3}}{1} = 1.7321^-$$

Now that we have learned something of the nature of the trigonometric functions, let us return to a consideration of our original problem involving the height of a tree.

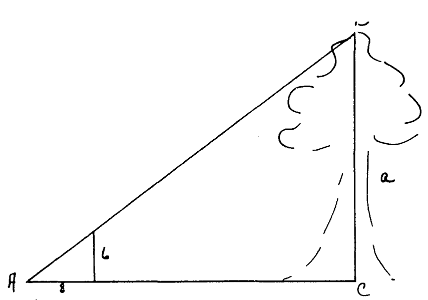

Figure 11

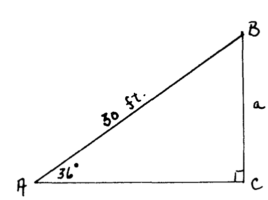

Figure 12

Using a transit (essentially an arrangement of telescope, and horizontal and vertical protractors), the surveyor would measure the angle at A (Fig. 11) by sighting the foot and top of the tree. He would find the angle to be about $36^0\ 52'$. Then he has the relationship

$$\frac{a}{40} = \tan 36^0\ 52'.$$

From the tables he finds $\tan 36^0\ 52' = .75$
Whence,

$$\frac{a}{40} = .75$$

and

$$a = 40 \times .75 = 30$$

A variety of problems involving right triangles can be solved similarly. We cite a few interesting examples. (Note—Four-place tables were used in the solutions of these examples.)

I. How far up a wall will a 30-foot ladder (Fig. 12) reach if the foot makes an angle of 36^0 with the ground?

$$\frac{a}{30} = \sin 36^0$$

$$a = 30 \times 0.5878 \text{ or } 17.63 \text{ ft.}$$

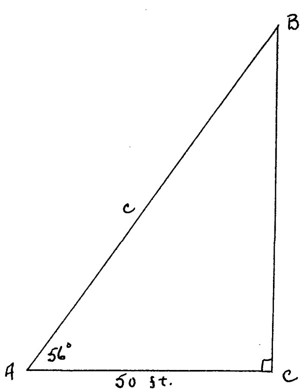

Figure 13

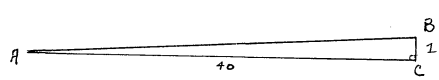

Figure 14

II. How long is the guy rope which supports a pole (Fig. 13), if the rope makes an angle of 56^0 with the ground and is fastened 50 ft. from the foot of the pole?

$$\frac{50}{c} = \cos 56^0$$

$$50 = c \times \cos 56^0 = 0.5592\, c$$

$$\frac{50}{0.5592} = c$$

whence

$c = 89.4$ ft.

III. What is the inclination of a plane (Fig. 14) which rises 1 foot in a horizontal distance of 40 feet?

$\tan A = 1/40$ or $.0250$

$A = 1^\circ\ 26'$

Unfortunately, the conditions of a problem do not always lend themselves to a solution by means of a right triangle. Finding the height of a mountain, for example, may lead to a situation like that in Fig. 15 in which side c, and the angles of elevation at A and B can be measured: Obviously, some new method will have to be derived to find h.

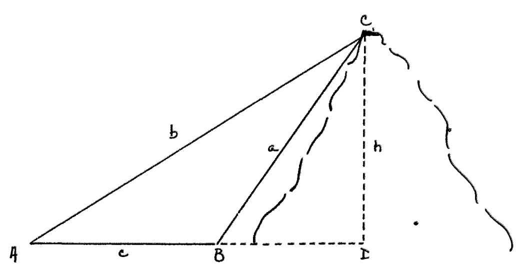

Figure 15

A nutcracker demonstrates the fact that, as an angle of a triangle changes, the side opposite that angle changes with it. Mathematically, we are concerned with the exact nature of this variation and it will be shown that the side increases or decreases in direct proportion with the sine of the opposite angle. From Fig. 15 we have, by the definition of the sine of an angle,

$$\frac{h}{b} = \sin A \quad \text{and} \quad \frac{h}{a} = \sin B$$

Whence

$h = b \sin A$ and $h = a \sin B$

Equating these two values of h, we have

$b \sin A = a \sin B$

Dividing both sides by $b \sin B$ gives

$$\frac{b \sin A}{b \sin B} = \frac{a \sin B}{b \sin B}$$

Cancelling out like terms in numerators and denominators, we obtain

$$\frac{\sin A}{\sin B} = \frac{a}{b}$$

Stated in words: *The sides of a triangle are proportional to the sines of the opposite angles.* In its complete form the Law of Sines may be written:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

Now let us proceed with the solution of the problem of Fig. 15. Suppose, upon measurement, the angles of elevation at A and B prove to be 34° and 50°, respectively, and side c measures 3000 ft. Then angle ABC is $180^\circ - 50^\circ$ or 130° and angle $C = 180^\circ - (130^\circ + 34^\circ) = 16^\circ$. Applying the Law of Sines, $\frac{a}{\sin A} = \frac{c}{\sin C}$, we have in this problem

$$\frac{a}{\sin 34^\circ} = \frac{3000}{\sin 16^\circ}$$

Multiplying both sides by $\sin 34^\circ$, and using a four-place table, we obtain

$$a = \frac{3000 \times \sin 34^\circ}{\sin 16^\circ} = \frac{3000 \times 0.5592}{0.2756} = 6087$$

In the right triangle BCD, angle B is 50° and $a = 6087$. Whence

$$\frac{h}{6087} = \sin 50^\circ$$

$$h = 6087 \times \sin 50^\circ = 6087 \times 0.7660 = 4663 \text{ ft.}$$

Were we required to find side b in the oblique triangle just considered, application of the Law of Sines would necessitate finding the value of the sine of angle ABC, that is, $\sin 130^\circ$. This suggests that we need to generalize the idea of trigonometric functions of angles to include angles greater than 90°. In fact we consider trigonometric functions of angles even greater than 180°. Although angles greater than 180° cannot be used in triangles, they are met in problems involving rotary motion.

Since the unit circle shows continuous changes in the functions as the angle increases toward 90°, we again turn to this device to build our concept of functions of angles greater than 90°. For convenience we adopt the mathematical convention of dividing the plane into four quadrants by means of a pair of perpendicular lines called axes, (see Fig. 16) and of regarding distances measured upward, and to the right, as positive, and those measured downward, and to the left, as negative. As before (in Fig. 7), PQ represents $\sin A$ and we generalize our definition of the sine to be the perpendicular distance to the horizon-

tal axis from the point at which the terminal side of the angle touches the unit circle.

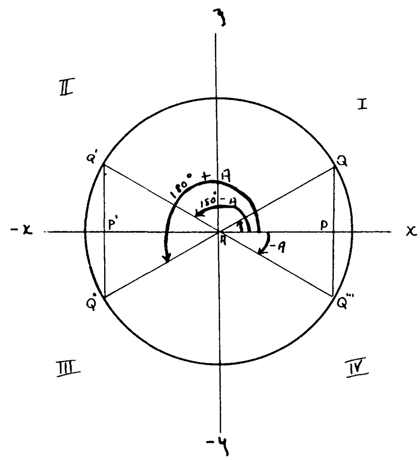

Figure 16

Then the lines corresponding to PQ in the other quadrants represent the sines of angles as follows:

$$P'Q' = \sin PAQ' = \sin (180^\circ - A)$$
$$P'Q'' = \sin PAQ'' = \sin (180^\circ + A)$$
$$PQ''' = \sin PAQ''' = \sin (360^\circ - A)$$

But lines PQ, $P'Q'$, $P'Q''$, and $PQ'''$are all equal in length. Some of them, however, differ in direction, thus

$$P'Q' = PQ$$
$$P'Q'' = -PQ$$
$$P'Q''' = -PQ$$

Therefore, it is possible to express the sines of angles in the second, third and fourth quadrants in terms of the sine of a related angle in the first quadrant. These relations follow:

Quadrant II	$\sin (180^\circ - A) = P'Q' = PQ = \sin A$
Quadrant III	$\sin (180^\circ + A) = P'Q'' = -PQ = -\sin A$
Quadrant IV	$\sin (360^\circ - A) = PQ''' = -PQ = -\sin A$

We list the following numerical examples (using a four-place table):

$$\sin 130^\circ = \sin (180^\circ - 50^\circ) = \sin 50^\circ = 0.7660$$

$$\sin 230^\circ = \sin (180^\circ + 50^\circ) = -\sin 50^\circ = -0.7660$$

$$\sin 310^\circ = \sin (360^\circ - 50^\circ) = -\sin 50^\circ = -0.7660$$

By means of the same device, similar relations can be derived for the cosine and tangent.

We now have at our command sufficient knowledge of trigonometry to appreciate and solve problems in some of the most important fields of applied mathematics. In navigation, one meets such problems as the following:

> A pilot wishes to make good a course in the direction 220° (with the northern direction). A 25 m.p.h. wind is blowing from 80°. If his air speed is 200 m.p.h., in what direction must he head the plane and what will be the ground speed?

By use of the sine law and a table of the values of the trigonometric functions one calculates the desired direction to be 215° 23′ and the desired speed to be 218.5 m.p.h., but we shall omit the details here.

A further use of trigonometry becomes apparent from a geometrical representation (graph) of the equation $y = \sin x$ on a pair of perpendicular lines as indicated in Fig. 17. When a value is assigned to x, the corresponding value of y can be determined from triangles or from tables. For example, if $x = 30^\circ$, then $y = \sin 30^\circ = .5$. Other pairs of values are shown below:

x	0°	30°	60°	90°	120°	150°	180°	240°	270°	300°	330°	360°
y	0	.5	.866	1	.866	.5	0	−.866	−1	−.866	−.5	0

Plotting these pairs of values and joining them with a smooth curve, the graph of $y = \sin x$ takes the following form:

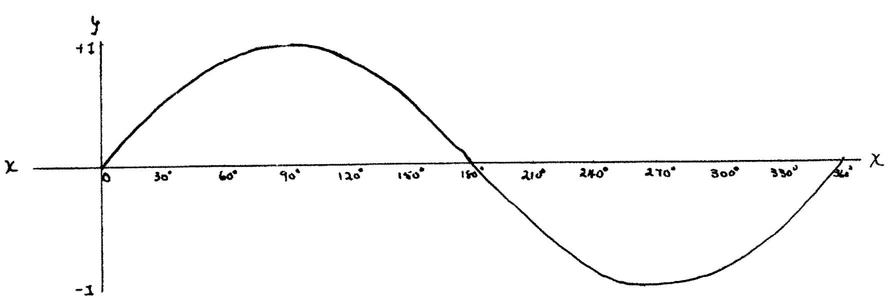

Figure 17

If we were to graph $y = \sin 2x$, the loops would be half as long, *i.e.*, there would be twice as many of them. Again, the graph of $y = 2 \sin x$ would differ from the graph of $y = \sin x$ in that it would vary from a maximum height of $+2$ to a minimum of -2, that is, the highest and lowest points of the curve would be twice as far apart as those on the graph of $y = \sin x$. In general, in the graph of $y = a \sin bx$, a is called the *amplitude* of the curve, and b the *frequency*. This terminology suggests that these curves are associated with the study of wave phenomena—light, sound and electricity. In fact, the curves traced by the needle on a speech-recording machine are sine curves.

While we have not, by any means, covered the solution of all types of triangles nor all the theory of trigonometry, we believe that this brief survey is sufficient to indicate that trigonometry has come a long way from the pseudoscience of astrology and that it occupies an honorable place among the branches of mathematics.

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